Algorithmic Thinking
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Sets

Definition 1 *A set is a collection of distinct items.*

A set can be described explicitly, as in
$$A = \{a, b, c, d\}$$
or
$$B = \{0, 1\}.$$
Alternatively, a set can be described symbolically using certain rules. For example,
$$C = \{n^2 : n \text{ is an integer}\}.$$
A special set is the *empty set*, denoted by $\emptyset$ or, alternatively, $\{\}$: it is the set that contains no elements.

We write $a \in A$ to denote that a is an element of A, and $a \notin A$ to denote that a is not an element of A.

Definition 2 *We say that set A is a subset of B, denoted by $A \subseteq B$, if every element in A is also an element of B. Or, using logic notation,*
$$\forall a \in A,\ a \in B.$$
For example, $\{a, b\} \subseteq \{a, b\}$; $\{0, 1\} \subseteq \{0, 1, 2, 3, 4, 5\}$. The empty set is a subset of every set, including the empty set itself:
$$\emptyset \subseteq \{1, 2, 3\} \qquad \emptyset \subseteq \emptyset \qquad \emptyset \subseteq \{\#, \%, \&\}.$$
If we want to emphasize that A is a proper subset of B, we use the notation $A \subset B$. This means that every element of A is also an element of B, and B contains at least one element that is not in A. For example, $\{a, b\}$ is not a proper subset of $\{a, b\}$, but $\{a, b\}$ is a proper subset of $\{a, b, c, d, e, f, 0, 1, 2, 3\}$.

Two sets A and B are equal, denoted by $A = B$, if they contain exactly the same elements.

Theorem 1 *$A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.*

1 Basic set operations

- Union: $A \cup B = \{a : a \in A \ \vee \ a \in B\}$.
- Intersection: $A \cap B = \{a : a \in A \ \wedge \ a \in B\}$.
- Difference: $A \setminus B = \{a : a \in A \ \wedge \ a \notin B\}$ (sometimes set difference is denoted $A - B$).
- Complement: $\overline{A} = U \setminus A$, where U is a universal set that is context-dependent.
- Cartesian product: $A \times B = \{(a, b) : a \in A \ \wedge \ b \in B\}$.
- Symmetric difference: $A \triangle B = (A \setminus B) \cup (B \setminus A)$.

For example, let $A = \{a, b\}$, $B = \{b, c\}$, and $C = \{0, 1\}$. Then,

- $A \cup B = \{a, b, c\}$, $A \cup C = \{a, b, 0, 1\}$, and $B \cup C = \{b, c, 0, 1\}$.
- $A \cap B = \{b\}$, $A \cap C = \emptyset$, and $B \cap C = \emptyset$.

- $A \setminus B = \{a\}$, $B \setminus A = \{c\}$, and $A \setminus C = A$.
- If $U = \{a, b, \ldots, z\}$, then $\overline{A} = \{c, d, \ldots, z\}$. However, if $U = \{a, b\}$, then $\overline{A} = \emptyset$.
- $A \times B = \{(a, b), (a, c), (b, b), (b, c)\}$ and $A \times C = \{(a, 0), (a, 1), (b, 0), (b, 1)\}$.

Notice that, by definition, $A \cup \emptyset = A$, $A \cap \emptyset = \emptyset$, $A \setminus \emptyset = A$, $A \times \emptyset = \emptyset$.

While the above operations are all binary, that is, take two sets as operands (except for the complement, which is unary), they can be extended to more than two sets. For example, for the three sets A, B, and C given above, we have

- $A \cup B \cup C = (A \cup B) \cup C = \{a, b, c, 0, 1\}$. Think about this as you first take the union of A and B, and then you take the union of the resulting set with C.
- $A \cap B \cap C = \emptyset$. Think about this as you first take the intersection of A and B, and then you intersect the resulting set with C.
- $A \setminus B \setminus C = (A \setminus B) \setminus C = \{a\}$.
- $A \times B \times C = \{(a, b, 0), (a, c, 0), (b, b, 0), (b, c, 0), (a, b, 1), (a, c, 1), (b, b, 1), (b, c, 1)\}$.

Definition 3 *The power set of a set A, denoted by $P(A)$ or 2^A, is the set of all subsets of A.*

For example, if $A = \{a, b\}$, then $P(A) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$. For $B = \emptyset$ we have $P(B) = \{\emptyset\}$.

From this definition we can establish that

$$A \subseteq B \text{ if and only if } A \in 2^B.$$

In plain English: A is a subset of B if and only if A is an element of the power set of B.

2 Set cardinality

Definition 4 *The cardinality of a set A, denoted by $|A|$, is the number of elements in A.*

For example, $|\{a, b, c\}| = 3$, $|\{\emptyset, \{a\}, \{b\}, \{a, b\}\}| = 4$, and $|\emptyset| = 0$. Using the definitions of the operations above, we can see that

- $|A \cup B| = |A| + |B| - |A \cap B|$.
- $|A \triangle B| = |A| + |B| - 2|A \cap B|$.
- $|A \setminus B| = |A| - |A \cap B|$.
- $|P(A)| = 2^{|A|}$.
- $|A \times B| = |A| \cdot |B|$.

3 Exercises

- Give an example of two non-empty sets whose union equals their intersection.
- Give an example of two sets whose cartesian product contains exactly three elements.
- How many subsets of set $A = \{a, b, c, d, e\}$ are there? List them.